Fuzzy Control

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1. Fuzzy Analysis

Definition 1: Fuzzy Set & Fuzzy Subset

A *fuzzy subset* A of a *universe of discourse* (or *support*) U is characterized by a *membership function* $\mu_A : U \to [0,1]$ which associates with each element x of U a number $\mu_A(x)$ in the interval $[0,1]$ which represents the grade of membership of x in A.

- $\mu_A(x) = 1$: x is a full member of A
- $\mu_A(x) \in (0,1)$: x is a part member of A
- $\mu_A(x) = 0$: x is not a member of A

B is a *fuzzy subset* of A if and only if $\mu_B(x) \le \mu_A(x), \quad \forall x \in U$.

A *fuzzy singleton* is a fuzzy set whose support is a single point in U.

Example 1:

Consider a fuzzy subset A with 5 elements whose membership functions are 0.7, 0.9, 1, 0.9. If this fuzzy subset A is defined as "around 10" on a scale from 8 to 12, it might be described as follows:

A = { 0.7|8, 0.9|9, 1|10, 0.9|11, 0.7|12 }

where {8, 9, 10, 11, 12} is called the *universe of discourse* or the *support*.

The fuzzy set

B = { 0.5|10, 0.8|11, 1|12 }

is a fuzzy subset of *A*.

The fuzzy set

A = { 0|8, 0|9, 0|10| 0.6|11, 0|12 }

is a fuzzy singleton.

Definition 2: Fuzzy Implication (Fuzzy Relation)

The *fuzzy implication* $R_{A\to B}$ is a fuzzy subset in the universe of discourse $X \times Y$ and is defined as

$$R_{A\to B} = \left\{ \left[(x,y), \mu_R(x,y) \right] \middle| \mu_R(x,y) = \mu_A(x) * \mu_B(y), \; x \in X, y \in Y \right\} \tag{1}$$

where the star operator "*" could be a *minimum* or *algebraic product*.

Remark 1:

- In the fuzzy literature, the term "fuzzy relation" and "fuzzy implication" are the same meaning. The term "fuzzy implication" is more precise since it is a *ordered relation* from X to Y, the term "relation" does not mean an ordered relation.
- If A and B are fuzzy sets then the *Cartesian product* $A \times B$ is a fuzzy implication. If X and Y are non-fuzzy then the Cartesian product has the conventional meaning, i.e. ordered pairs (x, y).

Definition 3: Fuzzy Composition

If R and S are fuzzy implications in $U \times V$ and $V \times W$, respectively. Then the *fuzzy composition* of R and R is a fuzzy implication in $U \times W$ and is defined as

$$R \circ S = \left\{ \left[(u,w), \sup_v \{ \mu_R(u,v) * \mu_S(v,w) \} \right], \; u \in U, v \in V, w \in W \right\} \tag{2}$$

where the star operator "*" could be a *minimum* or *algebraic product*.

Definition 4: Fuzzy Inference (Compositional Rule of Inference)

If a fuzzy implication $R_{A\to B}$ and a fuzzy set A' (the antecedent) are known, then the fuzzy set B' (the consequent) is inferred from A' by the following fuzzy composition

$$B' = A' \circ R_{A\to B} \tag{3}$$

Remark 2:

Note the differences among fuzzy implication, composition and inference; in particular between fuzzy implication and inference. The definition of fuzzy composition is used to define fuzzy inference which will be used intensively in fuzzy control, but not fuzzy composition.

Example 2:

If we assume that X and Y, the supports, are discrete and finite then fuzzy subsets A of X and B of Y can be represented as vectors of membership values of every elements $x \in X$ and $y \in Y$. Thus suppose X has 4 elements and Y has 3 elements and the fuzzy subsets A and B are given as

$$A = \begin{bmatrix} 0.3 & 0.6 & 1 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 1 & 0.7 \end{bmatrix}$$

so the fuzzy implication is

$$R_{A\to B} = \min(A, B) = \min\left(\begin{bmatrix} 0.3 \\ \mathbf{0.6} \\ 1 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 0.2 & 1 & 0.7 \end{bmatrix} \right) = \min \begin{bmatrix} (0.3, [0.2 \quad 1 \quad 0.7]) \\ (\mathbf{0.6}, [\mathbf{0.2} \quad \mathbf{1} \quad \mathbf{0.7}]) \\ (1, [0.2 \quad 1 \quad 0.7]) \\ (0.4, [0.2 \quad 1 \quad 0.7]) \end{bmatrix} \tag{4}$$

$$R_{A\to B} = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.6 \\ 0.2 & 1 & 0.7 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

Using the fuzzy inference of minimum operator, we check if $B = A \circ R_{A\to B}$,

$$A \circ R_{A\to B} = \sup_x \left(\min\{\mu_A(x), \mu_R(x,y)\} \right) = \sup_x \left(\min\left\{ \begin{bmatrix} 0.3 \\ 0.6 \\ 1 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.6 \\ 0.2 & 1 & 0.7 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \right\} \right) \tag{5}$$

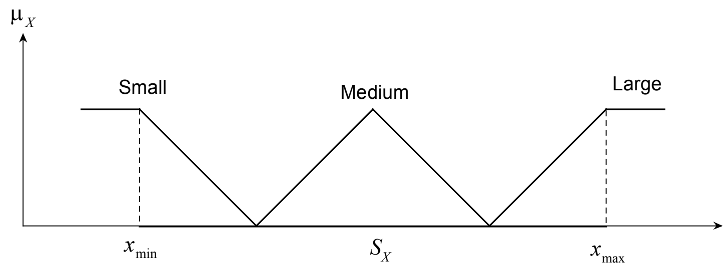
$$A \circ R_{A\to B} = \sup_x \left(\min \begin{bmatrix} (0.3, [0.2 \quad 0.3 \quad 0.3]) \\ (0.6, [0.2 \quad 0.6 \quad 0.6]) \\ (1.0, [0.2 \quad 1.0 \quad 0.7]) \\ (0.4, [0.2 \quad 0.4 \quad 0.4]) \end{bmatrix} \right) = \sup_x \begin{bmatrix} [0.2 \quad 0.3 \quad 0.3] \\ [0.2 \quad 0.6 \quad 0.6] \\ [0.2 \quad 1.0 \quad 0.7] \\ [0.2 \quad 0.4 \quad 0.4] \end{bmatrix} = \begin{bmatrix} 0.2 & 1 & 0.7 \end{bmatrix} = B$$

We have the same result for the fuzzy inference of algebraic product operator.

2. Fuzzy Control Parameters

2.1. Fuzzy Data Structure

Definition 5:



μ_X: membershipfunction, $S_X = [x_{\min}, x_{\max}]$: support *or* universe-of-discourse

Fig.1: Fuzzy parameters

From Fig.1 we have the following definitions

- **Fuzzy Set**

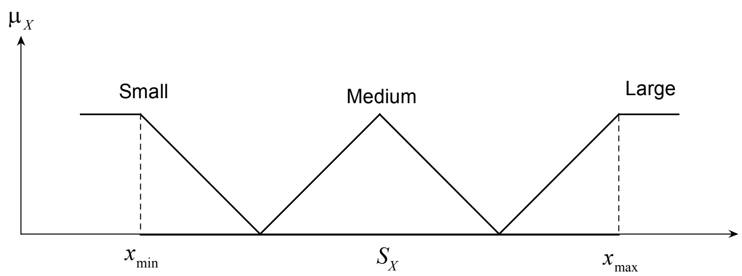


Fig.2: Fuzzy Set of *x*

- **Fuzzy Variable, Fuzzy Value**
  Fuzzy variable takes *fuzzy values*: small, medium and large which are defined as

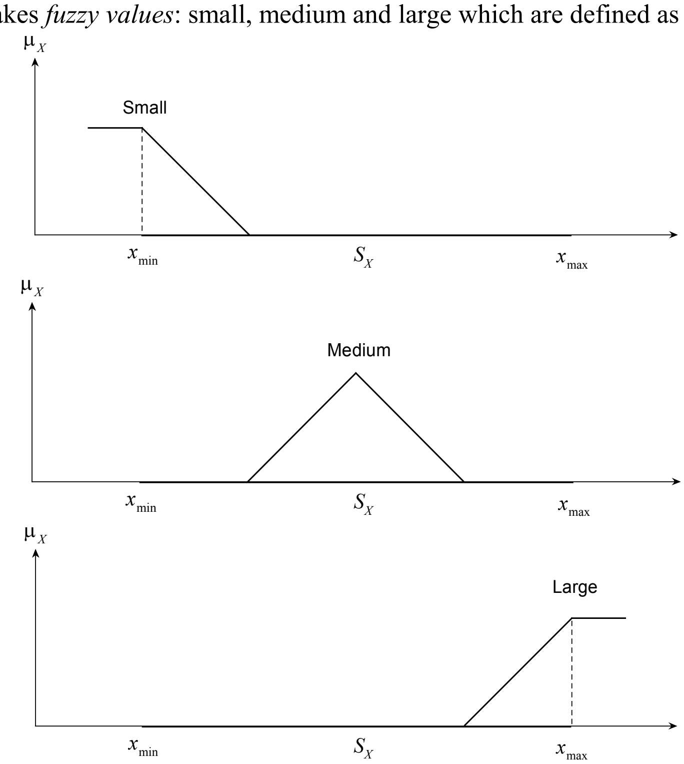


Fig.3: Fuzzy variable and fuzzy value

2.2. Fuzzy Reasoning

If an antecedent A' is a fuzzy singleton then the consequent B' can be determined by

$$A \to B \;\Rightarrow\; \min\{\mu_A(x_0), B\} \tag{6}$$

Thus it is not necessary to store the implication R_i's of the rule i-th or to compute R during the first phase of the algorithm. If x_0 and y_0 are known then for each rule $A_i \to B_i \to C_i$, $\mu_{A_i}(x_0)$ and $\mu_{B_i}(y_0)$ can be read off from the fuzzy subsets. Then the consequent of each rule C_i' is determined by

$$\boxed{C_i' = \min\{\min[\mu_{A_i}(x_0), \mu_{B_i}(y_0)], C_i\}} \tag{7}$$

Note that this is the *fuzzy inference* for the case of fuzzy singleton.

The consequent for the complete set of rules is determined by the following *fuzzy reasoning*

$$\boxed{C' = \max_i (C_i')} \tag{8}$$

then the corresponding *non-fuzzy* consequent is determined by the *mean of maxima defuzzification* method. This defuzzification method is preferred to the *centroid defuzzification* method since there are usually multiple of supports at the maximum membership.

Example 3:

For the same $R_{A\to B}$ given in the above example, let

$$A' = [0 \quad 1 \quad 0 \quad 0]$$

then from eq.(5), we have

$$A' \circ R_{A\to B} = \sup_x \left(\min\{\mu_A(x), \mu_R(x,y)\}\right) = \sup_x \left(\min\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.6 \\ 0.2 & 1 & 0.7 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \right\} \right) = R_{(2,:)}$$

from Eq.(4), we have

$$A' \circ R_{A\to B} = \min(0.6, B) = \min[\mu_A(x_0), B]$$

Remark 3:

- The fuzzy reasoning above still holds for the algebraic product operator instead of the minimum operator. The centroid defuzzification method may be preferred to the mean of maxima method for the algebraic product operator due to its smooth defuzzification.
- To apply the centroid method, for each rule C_i, Eq.(7) gives a fuzzy variable C_i' whose maximum membership function is μ_{C_i} at the corresponding support S_{C_i}. The mean of maxima method may be used to find this maximum. Finally for all rules, the non-fuzzy consequent can be determined by

$$\frac{\sum_i \mu_{C_i} S_{C_i}}{\sum_i \mu_{C_i}} \tag{9}$$

3. Fuzzy Control

3.1. Fuzzy Control Algorithm

Consider the following rulebase for an fuzzy variable error E and its change D

E \ *D*	NB	NS	ZO	PS	PB
NB	PB	PB	PS	PS	ZO
NS	PB	PS	PS	ZO	NS
ZO	PB	PS	ZO	NS	NB
PS	PS	ZO	NS	NS	NB
PB	ZO	NS	NS	NB	NB

Table.1: Rulebase for *E* and *D*

and their fuzzy sets

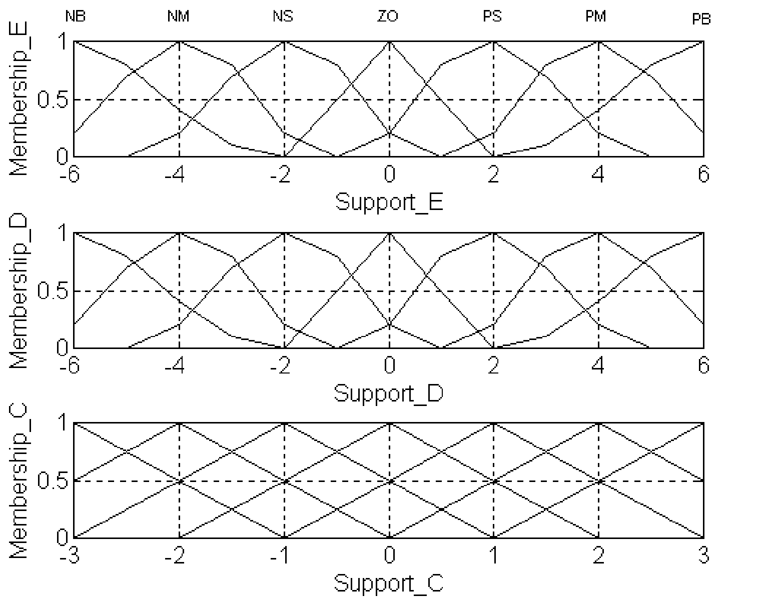


Fig.4: Fuzzy sets of *E, D* and *C*

For given error *e* and its change *d*, we will find a control *c* such that $e \to 0$. For each rule R_i in the rulebase, we have

$$R_i : \textit{If } e = FE_i \textit{ and if } d = FD_i, \textit{ then } c = FC_i \tag{10}$$

where FE_i, FD_i and FC_i take the fuzzy values such as *NB, NS, ZO, PS* and *PB.*

So due to rule R_i in Eq.(10) we have 2 types of value, namely real (crisp) values for both *E* and *D* and a fuzzy value for *C*

$$e = FE_i \Rightarrow \mu_{FE_i}(e)\text{: a real value} \tag{11}$$

$$d = FD_i \Rightarrow \mu_{FD_i}(e)\text{: a real value} \tag{12}$$

$$c = FC_i \Rightarrow FC_i\text{: a fuzzy value} \tag{13}$$

From these values, we can obtain a *required fuzzy value* for *C* using the fuzzy inference in Eq.(7)

$$C_i = \min\left[\mu_{FE_i}(e), \mu_{FD_i}(d), FC_i\right] \tag{14}$$

This is the required fuzzy value of *C* for the rule R_i only. The required fuzzy value of *C* for all the rules in the rulebase can be determined using the fuzzy reasoning in Eq.(8)

$$C = \max_i \left(C_i\right) \tag{15}$$

Finally, the real (crisp) value of the required control for the given pair (e, d) can be determined by a defuzzification, say mean-of-maxima

$$c(e, d) = dfz(C) \tag{16}$$

3.2. Fuzzy Control Implementations

Procedure 1: Fuzzy Control Procedure

For a given pair of error and its change (e, d)

Step 1: The fuzzy value of required control for the rule R_i

$$C_i = \min\left[\mu_{FE_i}(e), \mu_{FD_i}(d), FC_i\right] \tag{14}$$

Alternatively, minimum operator can be replaced by the algebraic-product operator.

Step 2: The fuzzy value of required control for *all* the rules in the rulebase

$$C = \max_i\left(C_i\right) \tag{15}$$

Step 3: The real value of required control for all the rules in the rulebase

$$c(e, d) = dfz(C) \tag{16}$$

The defuzzification could be the mean-of-maxima or the centroid method.

Remark 4: Conventional Fuzzy Control

For each pair of error and its change (e, d), the whole procedure above must run through all the rules in the rulebase. This execute time may exceed the sampling time, so the *continuous* error and its change (an infinite number of values) should be discretized into a finite number of values as *supports* of *E* and *D*, then a control matrix in a form of a look-up table will be determined in advance (off-line) using the procedure above. The rows and columns of the control matrix represent the supports of *E* and *D* respectively, and its elements represent the real values of the required control. For example, in a simulation below, the execute time is 7.41 seconds if using the on-line 7×7 control matrix, 20 seconds if using the on-line 9×9 control matrix and 0.77 sec if using the off-line control matrix. Note that the execute time above has been minimized because only the rules with non-zero membership for both *E* and *D* are considered. Therefore, if the execute time exceeds the sampling time, the error and its change should be discretized and a off-line control matrix should be determined in advance.

Remark 5: Simplified Fuzzy Control

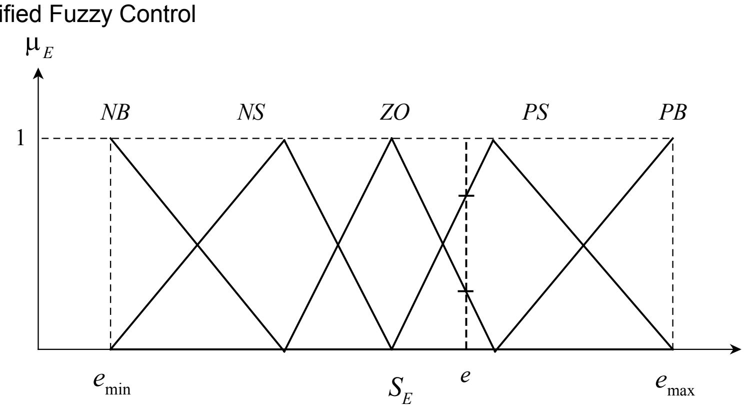


Fig.5: Fuzzy set of *E*

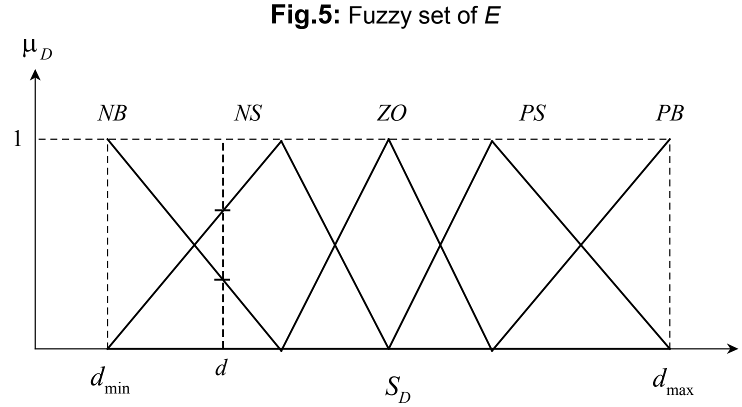


Fig.6: Fuzzy set of *D*

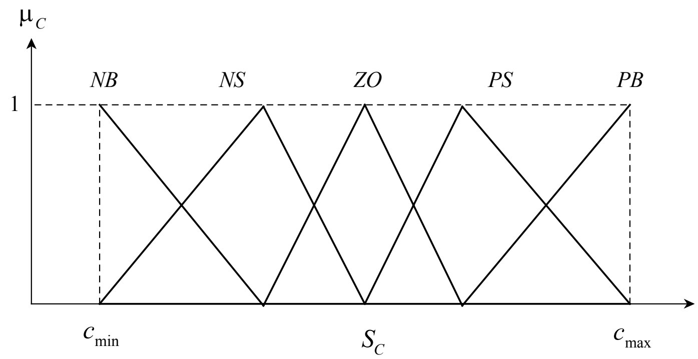


Fig.7: Fuzzy set of *C*

Consider fuzzy sets as in Fig.4 & 5, for any given pair (e,d) there are 4 rules atmost. In Eq.(15) of Step 2, only 4 rules are considered instead of all the rules. Furthermore, the real continuous values do not have to be discretized, so the associated resolution is out of question. For example, in a simulation below, the execute time is 3.13 seconds for $9\times 9=81$ rules in a simplified fuzzy control, 7.41 seconds for $7\times 7=49$ rules and 20 seconds for $9\times 9=81$ rules in a conventional fuzzy control.

4. Case Study

We will study the following system from Braae 1979

$$G(s)=\frac{400}{s^2+20s+400}$$

since some design details have been provided.

4.1. Original Design

Consider the following rulebase

E / S	NB	NM	NS	ZO	PS	PM	PB
NB	NM	NM	NM	NM	NS	ZO	ZO
NS	NM	NM	NM	NS	ZO	ZO	ZO
NS	NM	NM	NS	ZO	ZO	ZO	PS
ZO	NM	NS	ZO	ZO	ZO	PS	PM
PS	NS	ZO	ZO	ZO	PS	PM	PM
PS	ZO	ZO	ZO	PS	PM	PM	PM
PB	ZO	ZO	PS	PM	PM	PM	PM

Table.2: Rulebase for S (Sum_of_Error) and E (Error)

Fuzzy / Support	NB	NM	NS	ZO	PS	PM	PB
-6	1.0	0.2	0.0	0.0	0.0	0.0	0.0
-5	0.8	0.7	0.0	0.0	0.0	0.0	0.0
-4	0.4	1.0	0.2	0.0	0.0	0.0	0.0
-3	0.1	0.8	0.7	0.0	0.0	0.0	0.0
-2	0.0	0.2	1.0	0.0	0.0	0.0	0.0
-1	0.0	0.0	0.8	0.5	0.0	0.0	0.0
0	0.0	0.0	0.2	1.0	0.2	0.0	0.0
1	0.0	0.0	0.0	0.5	0.8	0.0	0.0
2	0.0	0.0	0.0	0.0	1.0	0.2	0.0
3	0.0	0.0	0.0	0.0	0.7	0.8	0.1
4	0.0	0.0	0.0	0.0	0.2	1.0	0.4
5	0.0	0.0	0.0	0.0	0.0	0.7	0.8
6	0.0	0.0	0.0	0.0	0.0	0.2	0.1

Table.3: Fuzzy Sets for S (Sum_of_Error) and E (Error)

Note that the graphic representations of these fuzzy sets is in Fig.4 above.

Fuzzy Support	NB	NM	NS	ZO	PS	PM	PB
-3	1.0	0.5	0.0	0.0	0.0	0.0	0.0
-2	0.5	1.0	0.5	0.0	0.0	0.0	0.0
-1	0.0	0.5	1.0	0.5	0.0	0.0	0.0
0	0.0	0.0	0.5	1.0	0.5	0.0	0.0
1	0.0	0.0	0.0	0.5	1.0	0.5	0.0
2	0.0	0.0	0.0	0.0	0.5	1.0	0.5
3	0.0	0.0	0.0	0.0	0.0	0.5	0.1

Table.4: Fuzzy Sets for C (Control)

The gain for E, S and C are

$$k_e = 6, \quad k_s = 48, \quad k_c = 2$$

The control matrix is

E / S	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-6	-2	-2	-2	-2	-2	-2	-2	-1	-1	0	0	0	0
-5	-2	-2	-2	-2	-2	-2	-2	-1	-1	0	0	0	0
-4	-2	-2	-2	-2	-2	-2	-1	0	0	0	0	0	0
-3	-2	-2	-2	-2	-2	-1	-1	0	0	0	0	0	0
-2	-2	-2	-2	-2	-1	-1	0	0	0	0	0	1	1
-1	-2	-2	-2	-1	-1	-1	0	0	0	0	0	1	1
0	-2	-2	-1	-1	0	0	0	0	0	1	1	2	2
1	-1	-1	0	0	0	0	0	1	1	1	2	2	2
2	-1	-1	0	0	0	0	0	1	1	2	2	2	2
3	0	0	0	0	0	0	1	1	2	2	2	2	2
4	0	0	0	0	0	0	1	2	2	2	2	2	2
5	0	0	0	0	1	1	2	2	2	2	2	2	2
6	0	0	0	0	1	1	2	2	2	2	2	2	2

Table 5: Original Control Matrix

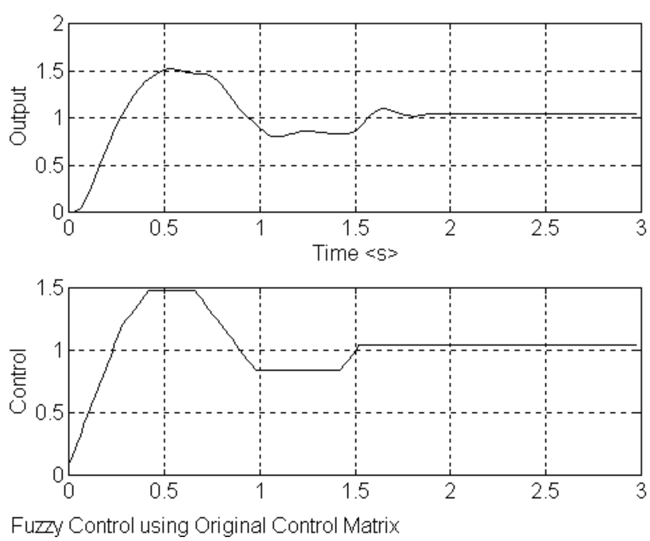


Fig.8: Fuzzy Control using Original Control Matrix in Braae *et al.* 1979.

4.2. Derivation of Control Matrix

Using the fuzzy data above, we obtain the following control matrix

E S	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
−6	−2	−2	−2	−2	−2	−2	−2	−1	−1	0	0	0	0
−5	−2	−2	−2	−2	−2	−2	−2	−1	−1	0	0	0	0
−4	−2	−2	−2	−2	−2	−2	−1	0	0	0	0	0	0
−3	−2	−2	−2	−2	−2	**−2***	−1	0	0	0	0	0	0
−2	−2	−2	−2	−2	−1	−1	0	0	0	0	0	1	1
−1	−2	−2	−2	**−2***	−1	−1	0	0	0	0	0	1	1
0	−2	−2	−1	−1	0	0	0	0	0	1	1	2	2
1	−1	−1	0	0	0	0	0	1	1	**2***	2	2	2
2	−1	−1	0	0	0	0	0	1	1	2	2	2	2
3	0	0	0	0	0	0	1	**2***	2	2	2	2	2
4	0	0	0	0	0	0	1	2	2	2	2	2	2
5	0	0	0	0	1	1	2	2	2	2	2	2	2
6	0	0	0	0	1	1	2	2	2	2	2	2	2

Table 6: Computed Control Matrix

* Note: these 4 entries are different from those in the original control matrix above.

With the same gains as above

$$k_e = 6, \quad k_s = 48, \quad k_c = 2$$

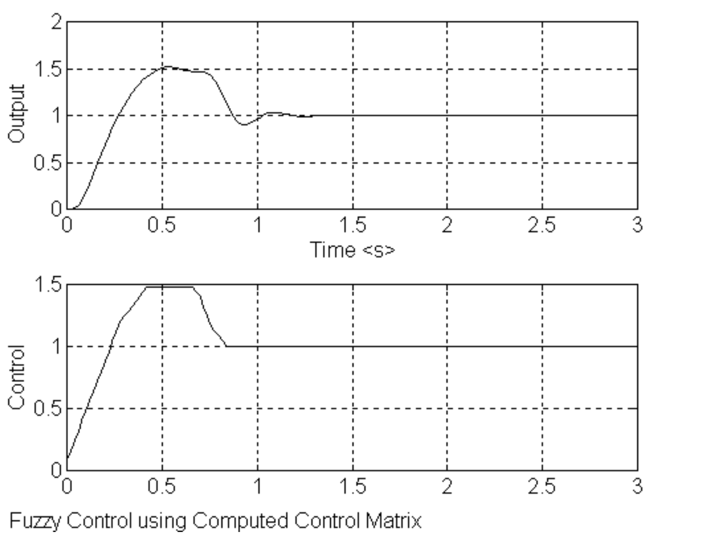


Fig.9: Fuzzy Contro using Computed Control Matrix

4.3. Simplified Fuzzy Control

Consider the following rulebase

S E	NB	NM	NS	nO	ZO	pO	PS	PM	PB
NB	PB	PB	PB	PM	PM	PS	PS	PO	ZO
NS	PB	PB	PM	PM	PS	PS	PO	ZO	NO
NS	PB	PM	PM	PS	PS	PO	ZO	NO	NS
NO	PM	PM	PS	PS	PO	ZO	NO	NS	NS
ZO	PM	PS	PS	PO	ZO	NO	NS	NS	NM
PO	PS	PS	PO	ZO	NO	NS	NS	NM	NM
PS	PS	PO	ZO	NO	NS	NS	NM	NM	NB
PS	PO	ZO	NO	NS	NS	NM	NM	NB	NB
PB	ZO	NO	NS	NS	NM	NM	NB	NB	NB

Table.7: Rulebase for E (Error) and S (Sum_of_Error)

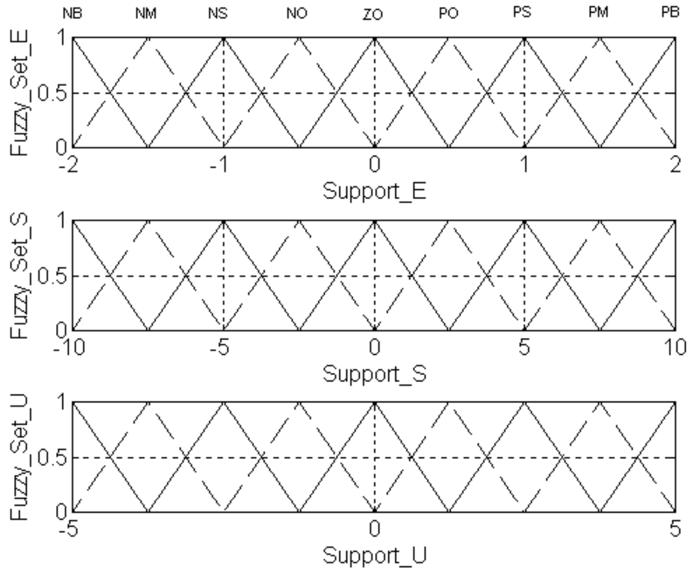


Fig.10: Fuzzy sets for E, S and U

The gain for E and S are

$$k_e = 0.05, \quad k_s = 12$$

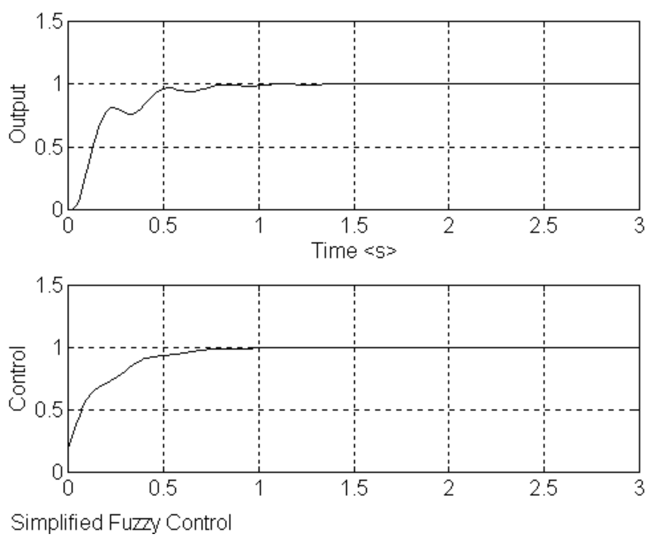


Fig.11: Simplified Fuzzy Control

4.4. Robustness of Fuzzy Control

Under 50% parameter variations

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{B}.u, \quad \mathbf{A} = \overline{\mathbf{A}} + \Delta\mathbf{A}, \quad \overline{\mathbf{A}} = \begin{bmatrix} 0 & 0 \\ -400 & -20 \end{bmatrix}, \quad |\Delta\mathbf{A}| \le \begin{bmatrix} 0 & 0 \\ 200 & 10 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 400 \end{bmatrix}$$

we have the following simulation

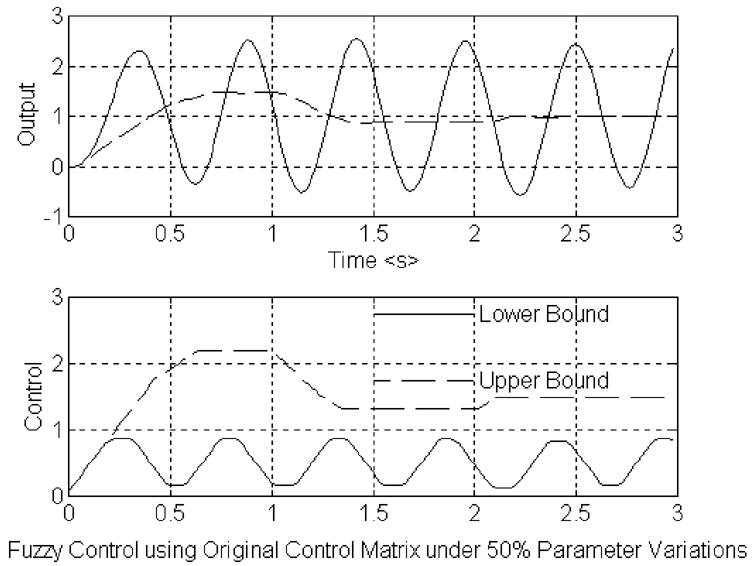


Fig.12: Fuzzy Control using Original Control Matrix under 50% Parameters Variation

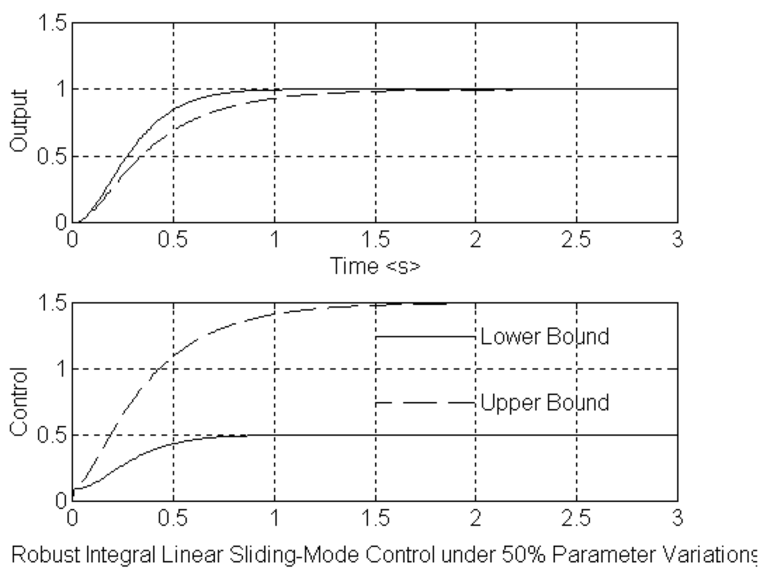


Fig.13: Robust Integral Linear Sliding-Mode Control under 50% Parameter Variations.

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